Section 1: The Nature of Numbers

M: *Help me understand why numbers behave the way they do.*

V: *Let's start with first principles. Pretend there is a one-dimensional universe that only our minds can enter. The indivisible units of space that make up this world are all junctions for special activities. They are infinite in number and align at precisely the same distance from each other. Like this:*

←••→

M: *It looks like a straight line of small black dots.*

V: *Don't be fooled by its simplicity. The more we poke around here - the more intriguing it becomes.*

M: *What is it like to spend time in a place like this?*

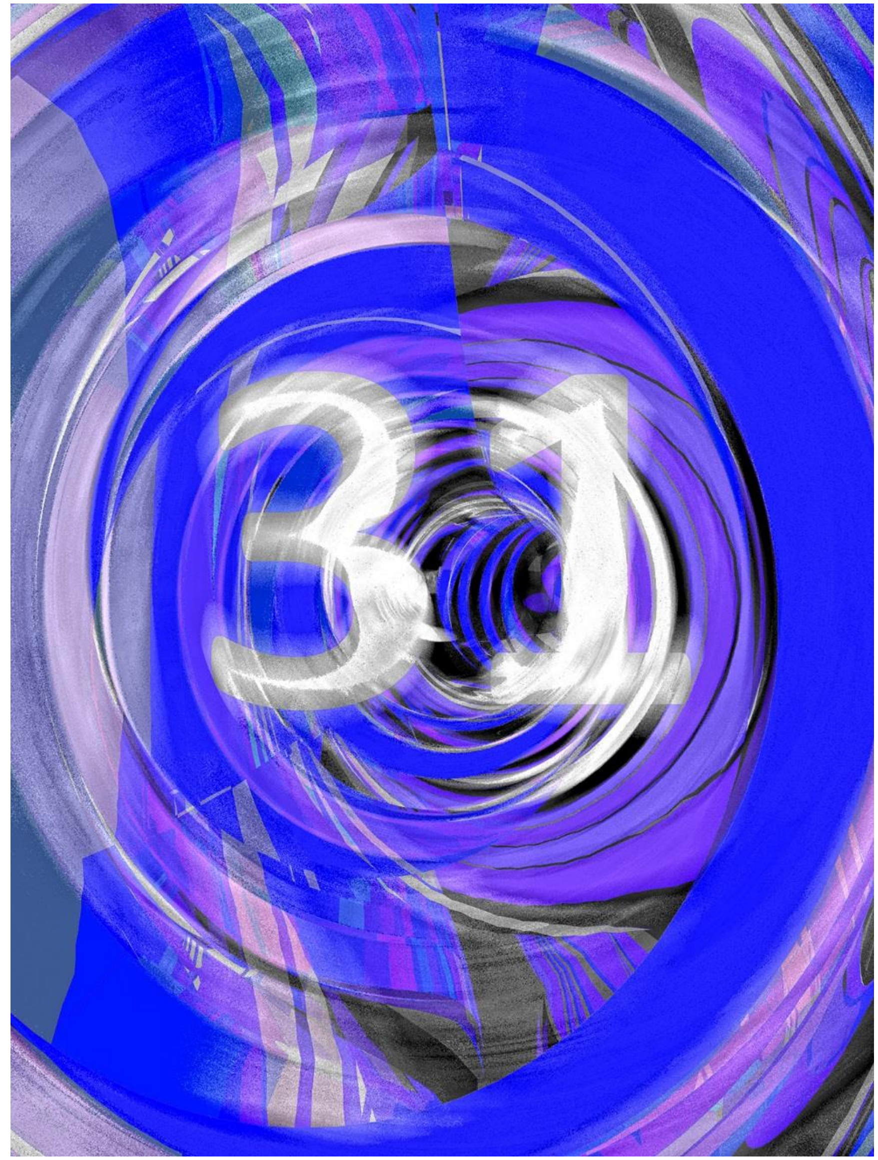
V: *At first, it feels very claustrophobic. It's like being in an incredibly narrow and long hallway with absolutely no headroom*

M: *It sounds like my first math class.*

V: *Don't be so quick to judge. With a little imagination, it can be a fun place to explore.*

M: *Okay, let's take a closer look.*

PZZT ZAP!



1.01 Positive, Negative and Zero

M: *Can we move in this world?*

V: *Yes, we can inhabit one or multiple units of space and move back and forth, to and fro, and forward and backward. Or, we can stay put and not move at all.*

Text Message to a Friend:

Dear Friend,
Having an exhausting time here in 1-D World. Yesterday, we moved around a great deal. We went this way and then that way. However, we soon tired and did nothing for a while. Last night, we relocated to where we are now. Will keep you posted if we come across anything out of the ordinary.

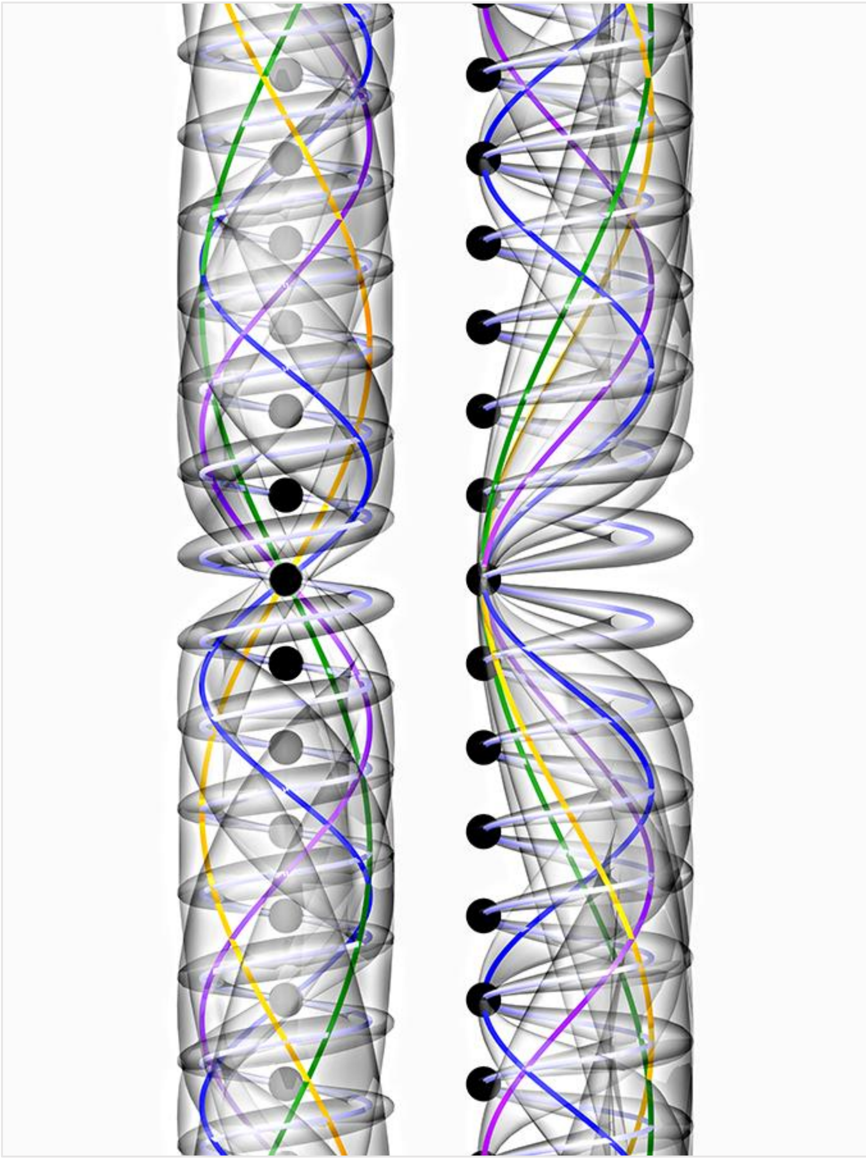
V: *In other words, we can move in a positive (+) direction or the opposite, in the negative (-) direction, or not at all by remaining neutral at our point of origin (0). In the example below, Arrows pointing right indicating we are moving in the positive direction and those aiming left in the negative direction.*

← ●←●←●←●←●←●←●←0→●→●→●→●→●→●→● →

M: *But everything looks the same here. How do we know where we have been or how far we have gone? And after a while, my text messages to friends back home all sound the same.*

V: *But every unit of space here isn't the same. Every point is unique.*

M: *Alright, then show me their differences, so I know where I am. Show me some landmarks.*



1.05 Multiples and Divisors

V: *As we just learned, integers exist in our one-dimensional universe at every juncture of the number line. And by the use of the Base-10 numbering system, we can now distinguish one integer from another.*

M: *Other than the names we coined for them, is there anything unique about our integers?*

V: *Yes. There exists a hidden network that links integers together. How these connections operate is unique to every integer. And like fingerprints, they can be used to identify an integer.*

M: *How does this concealed network work?*

V: *The system of connections has two interrelated components called multiples and divisors.*

Multiples are the outward expression of an integer. In our one-dimensional universe, an integer's multiples reach into infinity in both the positive and negative directions.

M: *How do the multiples multiply?*

V: *Every string of multiples starts at zero. The multiples progress in a series of increments equal to the number of units the integer is from zero. In this sense, the integer embeds a copy of itself into the places its multiples occupy.*

M: *And these embedded copies are the links that connect integers?*

V: *Yes. They are called divisors.*

M: *Give an example.*

V: *For example, the multiples of three are spaced every three places from zero. Three displayed as an embedded place holder, the multiples of three look like this:*

←3 ● ● 3 ● ● 3 ● ● 3 ● ● 0 ● ● 3 ● ● 3 ● ● 3 ● ● 3→

M: *Aren't the divisors embedded in spaces already occupied by other integers. For example, the multiples of three fall on these integers:*

←-12 ● ●-9 ● ●-6 ● ●-3 ● ● 0 ● ● 3 ● ● 6 ● ● 9 ● ● 12→

V: *Yes. And, referring to multiples in this way helps to distinguish an integer's multiples from each other.*

M: *Notice how the integers linked by the multiples of three are three times greater than the multiples of one.*

V: *Yes. The multiples of all integers are proportional to the multiples of one.*

Also, the only common point the multiples all integer share is zero.

M*: What are some unique characteristics of divisors?*

V: *Divisors are the accumulated place holders embedded at a unit by the multiples of integers. Unlike multiples, divisors are finite.*

As an example, the positive divisors left on 12 by the multiples of all integers are 1, 2, 3, 4, 6, and 12.

The multiples of one are embed into every integer. So, the number one is a divisor of every integer.

M: *Is there an even distribution of divisors throughout all integers?*

V: *No there isn't. That's what makes integers quirky and intriguing.*

For example, integers that are multiples of 12, especially those ending in zero, get the most action when it comes to collecting divisors. Their popularity as a place to visit among integers makes them divisible into a variety of whole numbers. That is why we apportion 12 inches in a foot, 24 hours in a day, 60 minutes in an hour, and 360 degrees in a circle.

M: *If the integers that are multiples of 12 are so popular, are their antisocial integers that have the least amount of divisors.*

V: *Yes, integers that are only divisible by themselves and one. Mathematicians have been intrigued and baffled by these numbers for thousands of years. They are called prime numbers.*

1.08 A Closer Look at Multiples and Divisors

M: *I'm having a hard time visualizing all the connections going on in our one-dimensional number line. With their multiples and divisors, a lot seems to be going on between integers.*

V: *Let's try looking at integers and their multiples in a different way - as a string of divisors. We start by separating and isolating each integer's string of multiples from our one-dimensional number line and stacking them on top of each other in rows, like the table to the right.*

M: *And the black dots represent the number of spaces between the divisors?*

V: *That's right. Notice that on the top row the multiples of 1 leave a divisor of one on every unit of our number line; on the second row the multiples of 2, every other place; and on the third row the multiples of 3 every three place and so on.*

M: *These rows of divisors go on for infinity but I can see enough of their relationships represented here to start to see patterns emerge.*

0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	•	2	•	2	•	2	•	2	•	2	•	2	•	2	•	2	•	2	•	2	•	2
0	•	•	3	•	•	3	•	•	3	•	•	3	•	•	3	•	•	3	•	•	3	•
0	•	•	•	4	•	•	•	4	•	•	•	4	•	•	•	4	•	•	•	4	•	•
0	•	•	•	•	5	•	•	•	•	5	•	•	•	•	5	•	•	•	•	5	•	•
0	•	•	•	•	•	6	•	•	•	•	•	6	•	•	•	•	•	6	•	•	•	•
0	•	•	•	•	•	•	7	•	•	•	•	•	•	7	•	•	•	•	•	•	7	•
0	•	•	•	•	•	•	•	8	•	•	•	•	•	•	•	8	•	•	•	•	•	•
0	•	•	•	•	•	•	•	•	9	•	•	•	•	•	•	•	•	9	•	•	•	•
0	•	•	•	•	•	•	•	•	•	10	•	•	•	•	•	•	•	•	•	10	•	•
0	•	•	•	•	•	•	•	•	•	•	11	•	•	•	•	•	•	•	•	•	•	11
0	•	•	•	•	•	•	•	•	•	•	•	12	•	•	•	•	•	•	•	•	•	•
0	•	•	•	•	•	•	•	•	•	•	•	•	13	•	•	•	•	•	•	•	•	•
0	•	•	•	•	•	•	•	•	•	•	•	•	•	14	•	•	•	•	•	•	•	•
0	•	•	•	•	•	•	•	•	•	•	•	•	•	•	15	•	•	•	•	•	•	•
0	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	16	•	•	•	•	•	•
0	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	17	•	•	•	•	•
0	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	18	•	•	•	•
0	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	19	•	•	•
0	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	20	•	•
0	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	21	•
0	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	22

To determine the multiples of any given integer, multiply the integer individually by all the other integers.

For example: ...-4N, -3N, -2N, -N, 0, N, 2N, 3N, 4N..., where N equals some integer.

Notes:

In the graphic to the right, the positive multiples of 1 to 22 are aligned horizontally with their zeros on their left and stacked in rows in descending order.

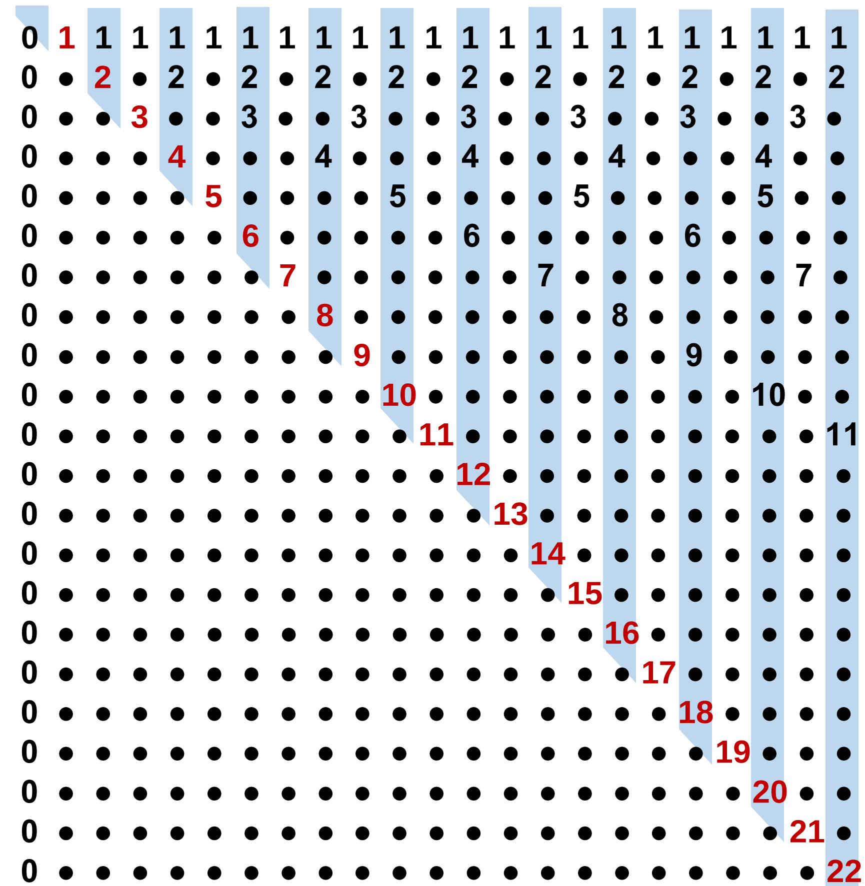
1.09 Seeing the Divisors in Multiples

***V:** Okay, now let's look at the graph to the right as a series of vertical relationships. Notice that each column contains all the divisors of the integers highlighted in red.*

***M:** It's interesting to see the underlying visual relationships between multiples and divisors. Their spacing has an elegance.*

***V:** Without the spacing, it is difficult to sense the balance and harmony between the same multiples and divisors. See below.*

Integer	Divisor(s)
1	1
2	1, 2
3	1, 3
4	1, 2,
5	1, 5
6	1, 2, 3, 6,
7	1, 7
8	1, 2, 4,8
9	1, 3, 9
10	1, 2, 5, 10
11	1, 11
12	1, 2, 3, 4, 6, 12
13	1, 13
14	1, 2, 7, 14
15	1, 3, 5, 15
16	1, 2, 4, 8, 16
17	1, 17
18	1, 2, 3, 6, 9, 18
19	1, 19
20	1, 2, 4, 5, 20
21	1, 3, 7, 21
22	1, 2, 11, 22



Notes:

To determine the positive divisors of some integer N, divide N by all integers between 1 and N. Any resulting whole numbers are divisors.

For example: ...N/1, N/2, N/3, N/4...N/N, where N equals a given integer.

Section 2: **From Two- to Three-Dimensions**

M: *So now that we know our way around our one-dimensional universe and the nature of numbers, what can we do with it?*

V: *How about we make something three-dimensional using what we have learned about integers and their multiples.*

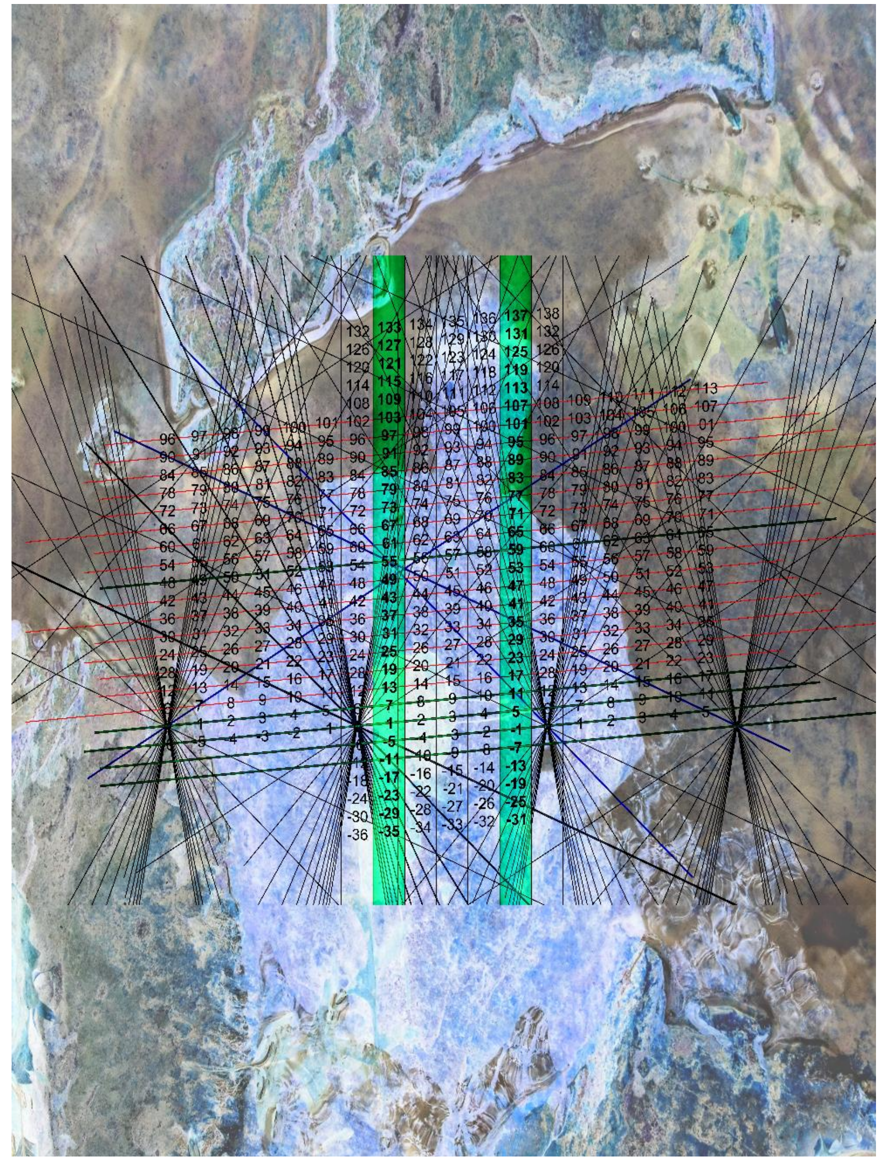
M: *Will there be any other patterns?*

V: *Yes but I'm still trying to make sense of it. Maybe you can help me understand what's going on.*

M: *I like a good puzzle.*

V: *We'll start by laying out our integer numbers line in two-dimensions and then figure out how to align them in three-dimensions.*

M: *Let's do it.*



2.01 Stacking the Number Lines

***V:** Okay, let's see what happens when we play with our one-dimensional number line. Let's start by making an infinite amount of them. We can then align them together to form a two-dimensional plane.*

***M:** You mean stack them on top of each other, like in Figure 2?*

***V:** Yes. Let's take notes of some of the patterns we find.*

Notes:

Rows of integers can be stacked vertically, filling an infinite two-dimensional plane in all directions.

All negative integers are to the left of the column of zeros and all positive integers to the right.

Moving horizontally on any row, the integers from number lines.

Moving diagonally in either direction, over one and up one, the integers also form number lines.

Moving vertically, the integers in the columns neither increase nor decrease.

Figure 2, number lines of integers stacked on top of each other:

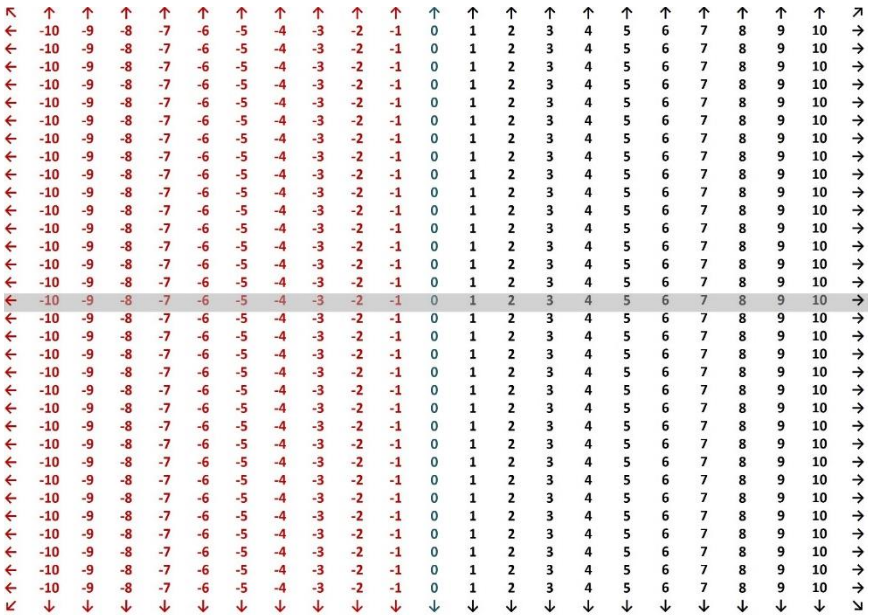


Figure 1

2.03 A Table of Six Columns

M: *Wow. This graphic makes the Table of Six Columns stand out. And just for fun, I colored the prime numbers blue. Notice how they fall in only specific columns.*

V: *Let's take a closer look at the Table of Six Columns. We'll name the individual columns that make up the table Columns A, B, C, D, E, and F. See our notes below.*

								A	B	C	D	E	F							
83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100	101	102	103
77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97
71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91
65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85
59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79
53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73
47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67
41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61
35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55
29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49
23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
-19	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1
-25	-24	-23	-22	-21	-20	-19	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5
-31	-30	-29	-28	-27	-26	-25	-24	-23	-22	-21	-20	-19	-18	-17	-16	-15	-14	-13	-12	-11
-37	-36	-35	-34	-33	-32	-31	-30	-29	-28	-27	-26	-25	-24	-23	-22	-21	-20	-19	-18	-17
-43	-42	-41	-40	-39	-38	-37	-36	-35	-34	-33	-32	-31	-30	-29	-28	-27	-26	-25	-24	-23
-49	-48	-47	-46	-45	-44	-43	-42	-41	-40	-39	-38	-37	-36	-35	-34	-33	-32	-31	-30	-29
-55	-54	-53	-52	-51	-50	-49	-48	-47	-46	-45	-44	-43	-42	-41	-40	-39	-38	-37	-36	-35
-61	-60	-59	-58	-57	-56	-55	-54	-53	-52	-51	-50	-49	-48	-47	-46	-45	-44	-43	-42	-41
-67	-66	-65	-64	-63	-62	-61	-60	-59	-58	-57	-56	-55	-54	-53	-52	-51	-50	-49	-48	-47
-73	-72	-71	-70	-69	-68	-67	-66	-65	-64	-63	-62	-61	-60	-59	-58	-57	-56	-55	-54	-53
-79	-78	-77	-76	-75	-74	-73	-72	-71	-70	-69	-68	-67	-66	-65	-64	-63	-62	-61	-60	-59
-85	-84	-83	-82	-81	-80	-79	-78	-77	-76	-75	-74	-73	-72	-71	-70	-69	-68	-67	-66	-65
-91	-90	-89	-88	-87	-86	-85	-84	-83	-82	-81	-80	-79	-78	-77	-76	-75	-74	-73	-72	-71
-97	-96	-95	-94	-93	-92	-91	-90	-89	-88	-87	-86	-85	-84	-83	-82	-81	-80	-79	-78	-77
-103	-102	-101	-100	-99	-98	-97	-96	-95	-94	-93	-92	-91	-90	-89	-88	-87	-86	-85	-84	-83

Figure 2

Notes:

The numbers captured in Columns A through F are a complete set of numbers, containing no like-numbers.

Columns A through F repeat every six columns and the numbers in the columns are now spaced six apart and can be described in the following ways:

- Column A = 6x+1*
- Column B = 6x+2 or 2(3x+1)
- Column C = 6x+3 or 3(2x+1)
- Column D = 6x+4 or 2(3x+2)
- Column E = 6x+5*
- Column F = 6x+6 or 6(x+1)

All integers in Columns B, D and F are evenly divisible by 2.

All integers in Columns C and F are evenly divisible by 3.

Because all the integers are divisible by 2 or 3, other than 2 and 3, no integer in Columns B, C, D, and F can be prime.

*All prime numbers, other than 2 and 3, are in either Column A or E. However, this does not mean that all the integers in Columns A and B are prime.

Notice that the integers located in Columns A and E are negative opposites of each other. In other words, all the positive numbers in Column A are negative numbers in Columns E, and vice versa. (This relationship will be significant later on when we limit potential prime numbers to integers found only in one of the two columns.)

2.04 Aligning Like-Numbers Horizontally

***V:** Now, let's align like-integers horizontally by adjusting all columns vertically.*

***M:** Does putting all number 66s on the same horizontal line make all other like integers align horizontally?*

***V:** That's right.*

***M**: Why is this important?*

***V:** By aligning like-numbers in this way allows us to properly link integers and their multiples in three-dimensions later on in Section 2.09.*

***M:** All the rows of integers in Figure 6 do seem to be less choppy now.*

Notes:

The stacked rows of integers are slightly angled, but each column remains vertically straight.

All positive numbers now are located above the horizontal line running through zeros, and all negative numbers are below.

***M:** I find it interesting how the orientation of positive and negative integers can so dramatically change. This sifting reminds me of the movement of a magnetic field.*

Figure 6, all like-numbers are aligned together on a separate horizontal line from the other like-numbers:

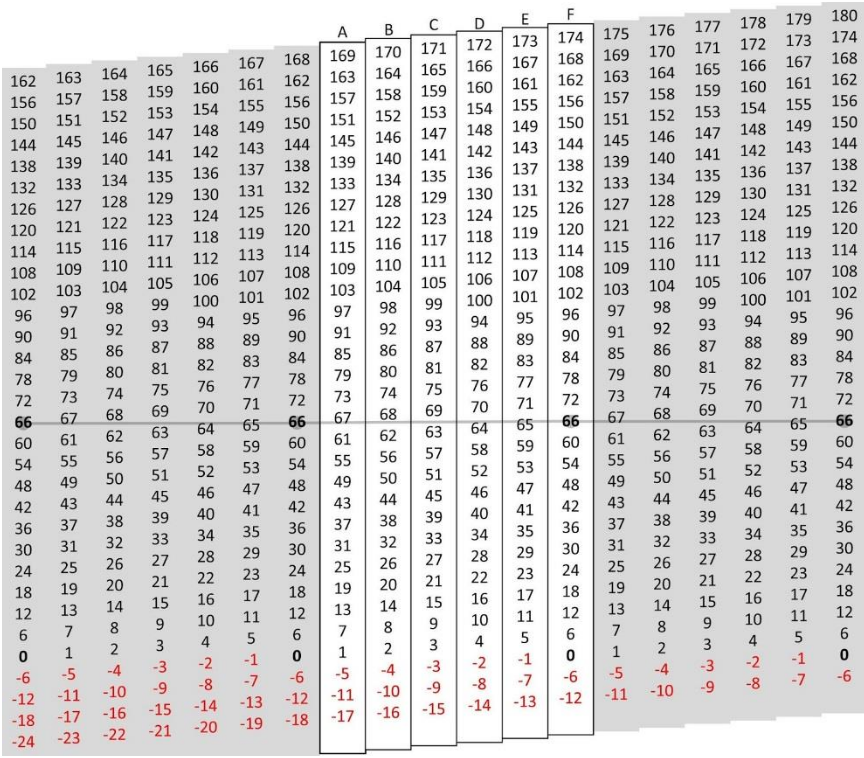


Figure 3

2.05 Inner Zeros

***V:** For now, our adjustments are complete.*

***M:** Great. Let's look for some integers and their multiples.*

***V:** Here's a pattern. Through points located at the two nearest zeros to Columns A through F, you can see straight lines of all the positive multiples of the 5 and 7.*

***M:** Cool. Let's see if there are any other multiples of integers cutting a path through our table of integers.*

Notes:

As shown in Figure 7, the positive multiples of 5 (5, 10, 15, 20, 25...) and 7 (7, 14, 21, 28, 35...) align.

Figure 7, starting at the two inner zeros, lines drawn through the positive multiples of 5 and 7:

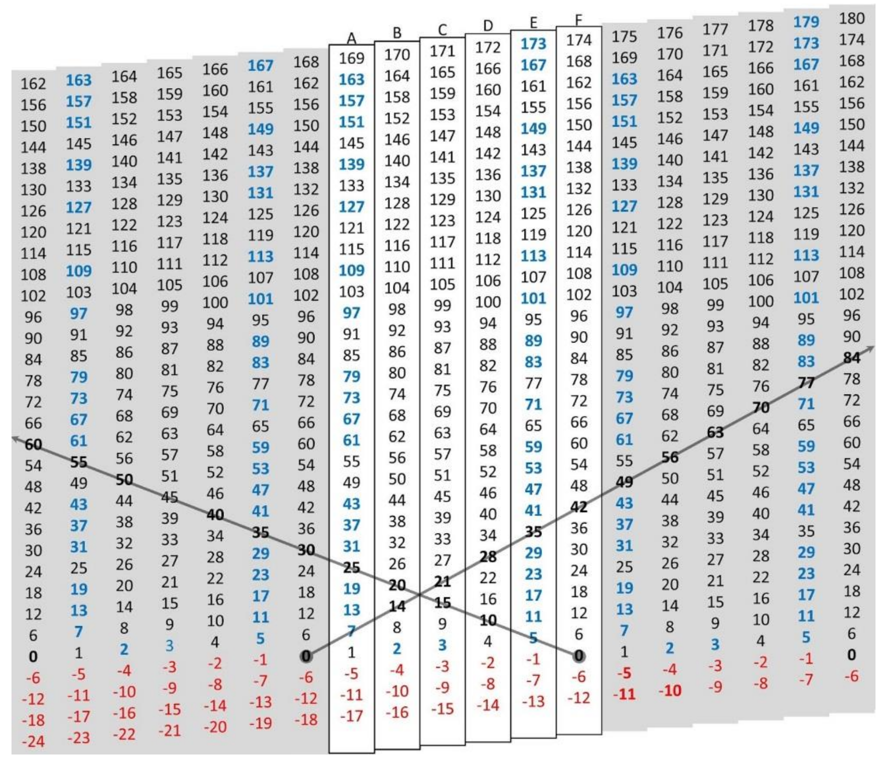


Figure 4

2.06 Radiating Multiples

V: *I can draw straight lines intersecting the two nearest zeros and the multiples of 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43 and 47.*

M: *Does this work for every integer in Columns A and E?*

V: *Yes, it does.*

M: *I'm glad we spent some time understanding integers and their chain of multiples in Section 1.*

V: *Remember that the point of origin for every chain of multiples starts at zero.*

Figure 8, lines can be drawn through the multiples of all the integers in Columns A and E:

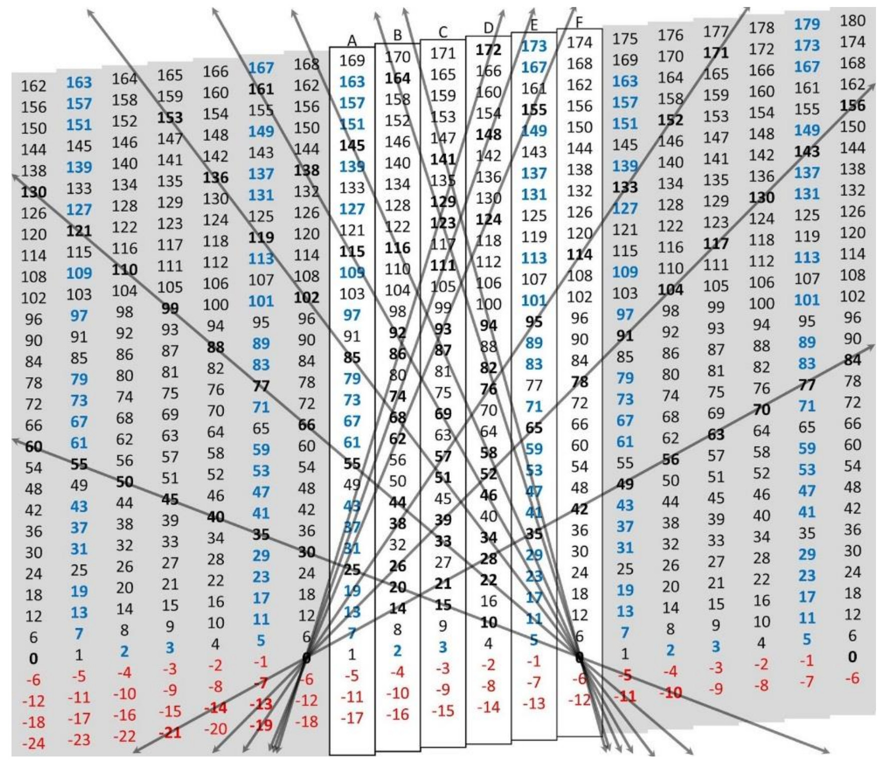


Figure 5

2.07 Outer Zeros

***V:** These two outer zeros are also radiating lines of multiples running through Columns A, B, C, D, E, and F.*

***M:** Does this apply for all zeros on our infinite plane?*

***V:** Yes, it does. The further out the zero in located from our columns of interest - the higher the radiating multiples cross the columns.*

Notes:

On the infinite two-dimensional plane, there is an infinite number of zeros.

All zeros radiate out the same patterns of multiples.

Figure 9, starting at the two outer zeros, lines can be dawn through the multiples in the repeating outer Columns A and E:

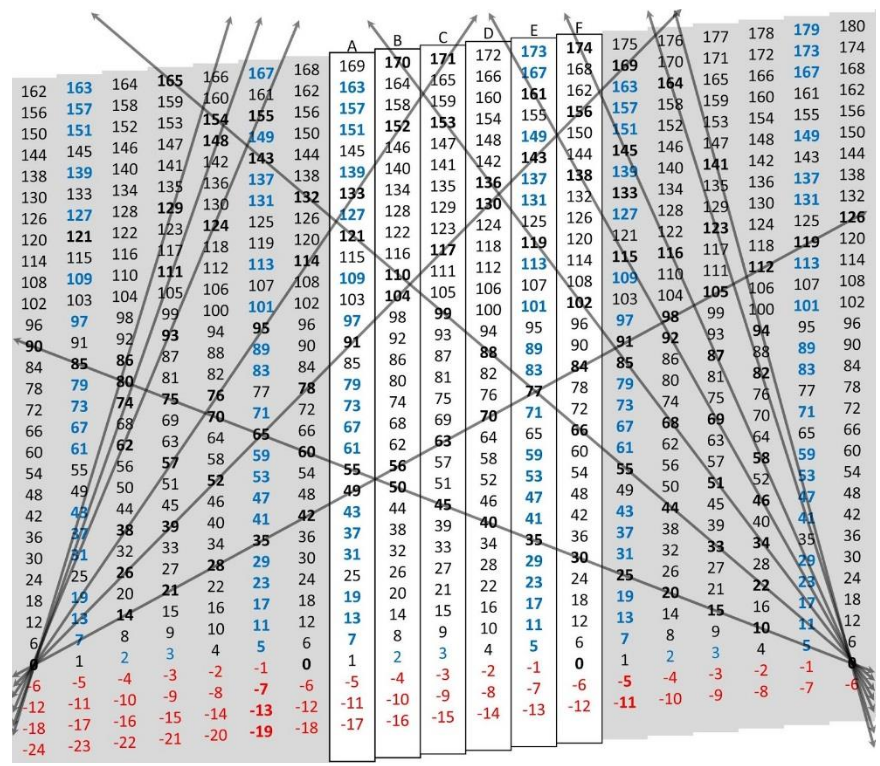


Figure 6

2.08 A Web of Multiples

M: *I'm having trouble keeping track of all the various multiples. It looks like we are playing Space Invaders.*

V: *Let's color code all the multiples of 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43 and 47 intersecting all zeros on the infinite plane.*

M: *I'm noticing that that lines of the same color run parallel with each other.*

V: *That's right. All multiples of the same integers, no matter what zero they emanate from, are equidistant and parallel with each other.*

Notes:

Figure 10 may look chaotic, but there is an underlining order. For example, all like-numbers and their multiples, represented by lines of the same color, intersect a zero and run parallel with each other. Also, the multiples of like-numbers are set at the same angle and are the same distance from each other.

Figure 10, all the multiples color-coded:

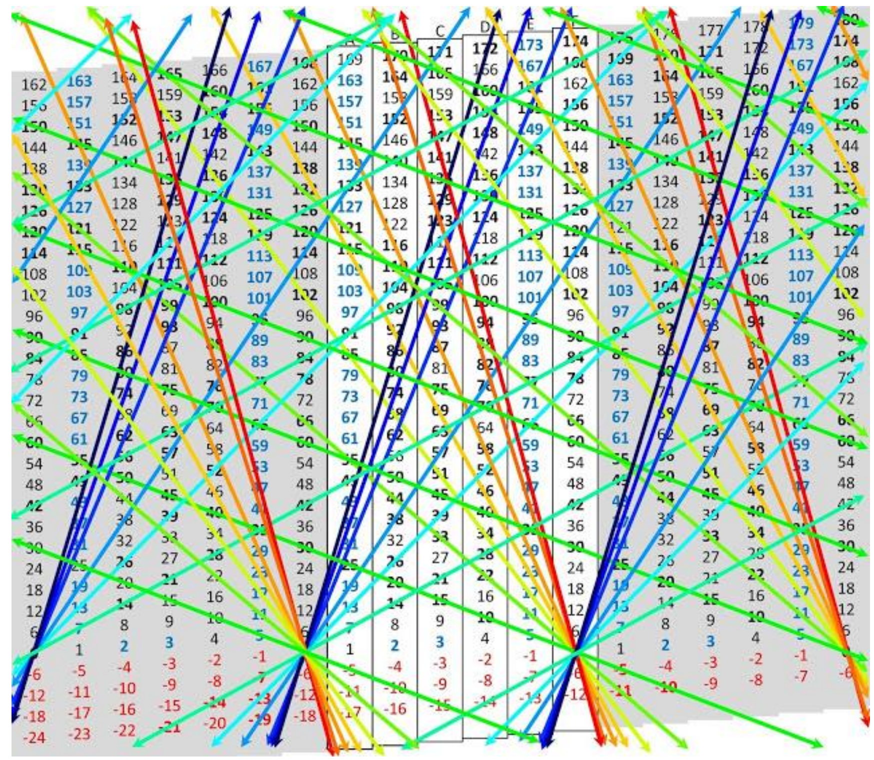


Figure 7

2.09 A Graphic Display of Primality

V: *Let's focus on the potential prime numbers in Columns A and E. Remember, other than 2 and 3, all prime numbers fall in these two columns. Figure 11 shows an enlargement of Column A in bold.*

M: *I see that the red parallel lines trace the multiples of one and that the blues lines indicate the multiples of 5, 7 and 11. The radiating black lines look they are coming from zero.*

V: *That's right. The red lines follow the multiples of one, intersecting all integers in numeric order. The black lines coming directly from zero are the first multiple of that integer.*

M: *I notice how the blue lines tracing the multiples of 5 and 11 intersect at 55. From Section 1, I know that 1, 5, 11 and 55 must be divisors left by all these multiple chains at 55. Therefore, 55 cannot be prime.*

V: *Also, we eliminated 49 as a prime.*

M: *So, 31, 37, 43, 61 and 67 must be prime.*

V: *That's right.*

Figure 11, a close-up detail from Figure 10:

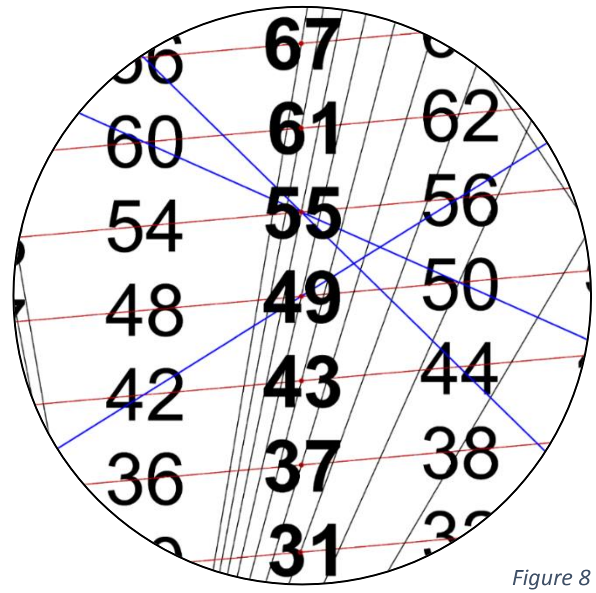


Figure 8

Notes:

Figure 11 is a blow-up from Figure 10. Numbers from Column A are in bold.

Notice that the red line representing the multiples of 1 intersect all integers and are parallel with each other.

Also, notice that the black lines radiate out from the same zero are intersecting the red dot of all the integers in Column A.

The numbers in Column A, 67, 61, 43, 37 and 31, are intersected by only one red parallel line and one black radiating line. Primes are only divisible by themselves and one.

The numbers 49 and 55, intersected by additional lines shown in blue, are not prime. A prime number is only divisible by itself and one.

2.10 From Two- to Three-Dimensions

M: *I thought we were going to make something three-dimensional out of these integers.*

V: *No problem. Just wrap Columns A, B, C, D, E and F into a three-dimensional cylinder. See Figure 12.*

M: *Wow. By rolling the table of integers into a cylinder, all the color-coded diagonal lines representing the multiples of integers now meet in perfect alignment making helixes.*

V: *The transition from two- to three-dimensions is far more dramatic when done with your own hands. To see for yourself, photocopy Figure 13 and connect the lines yourself.*

Notes:

In cylindrical form, there is only one zero, and the integers in Columns A and E and their multiples are contiguous following individual helical paths.

The integers that make up the cylinder are unique and do not repeat.

All helixes now share the same zero.

Figure 12, the multiples of the integers in Column A and E rolled into a cylinder.

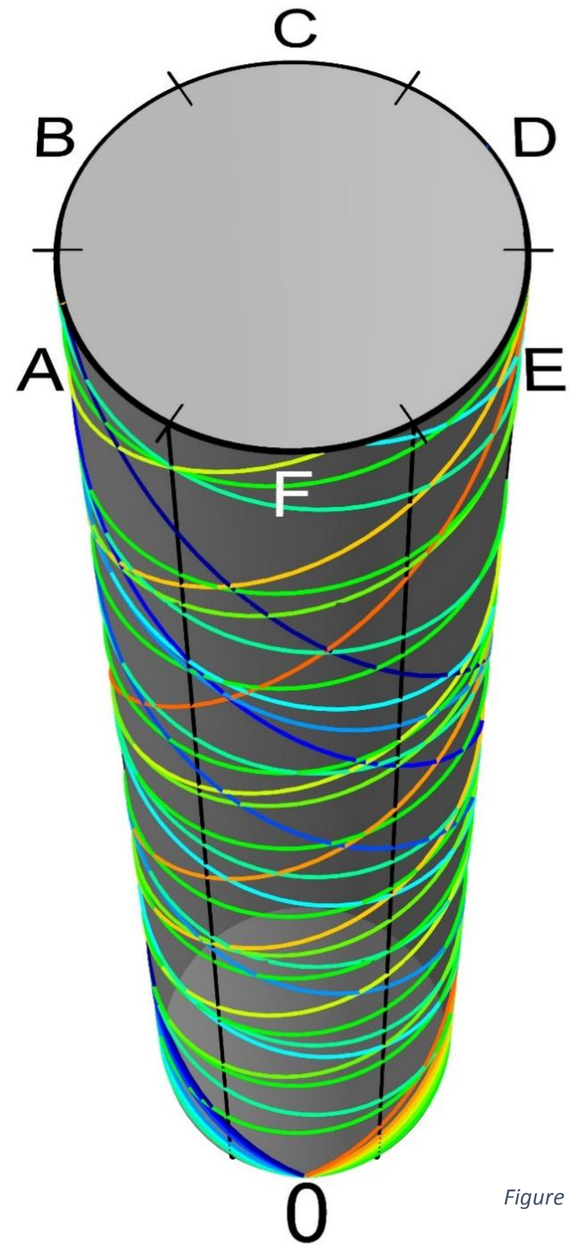


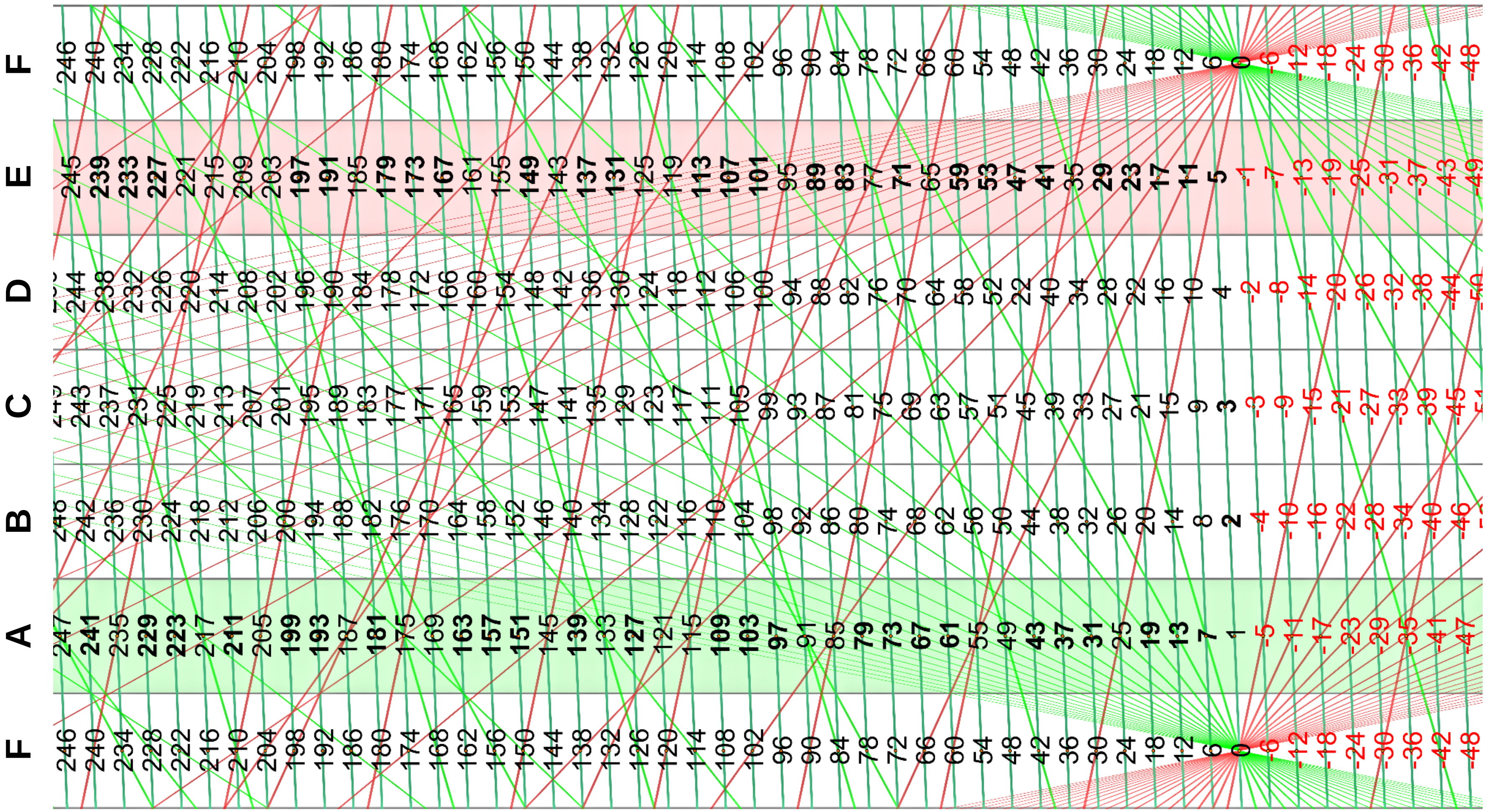
Figure 9

2.11 Prime Number Cylinder

M: *You can make your three-dimensional version by photocopying Figure 13, below.*

From your paper copy, cut out the rectangular graphic at the outer edges of the two F Columns. Roll the graphic into a cylinder around its longest axis. Overlap the two F Columns and tape or glue into place.

Figure 10



2.12 Conclusion

V: *This three-dimensional way of looking at integers and their multiples intrigues me.*

M*: This approach seems more natural than a two-dimensional plane or one-dimensional number line where we typically study prime numbers. And since we live in a three-dimensional world, maybe it has something fundamental to do with the universe around us.*

V*: It doesn't matter if there is an application to what we find here. In Pure Mathematics, we go where the patterns take us.*

M: *There is an esthetic to these shapes we discovered. I think I can base an art exhibition inspired by their beauty.*

V: *Putting these ideas out in public could lead to a greater understanding.*

M: *If I'm going to make these numeric relationships into art, then I need to find out more about this helical three-dimensional world of multiples.*

V: *It might be a good idea to start writing some of this down, too.*

M: *I've been taking notes in my sketchbook, but I think this idea requires a grander vision. I'll put together a narrative and print copies of our conversations in a book format. Then I can share this with the public at my art show.*

V: *There's still more to discover.*